8 A Category-Theoretic Reading of Peirce's System: Pragmaticism, Continuity and the Existential Graphs

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In this paper we present some central features of the Peircean system—the pragmaticist maxim, continuity and the existential graphs—which can be made precise and studied in a natural way by means of certain contemporary tools from the mathematical theory of categories. In the first section, we provide synthetic readings of the pragmaticist maxim, the Peircean continuum and the existential graphs. Though this material is well known, our approach is a novel one, and in this opening section we will lay the groundwork for our interpretation of the Peircean system. In the second section, we present a very short introduction to the basic category-theoretic concepts that we will need later on. The material, again, is well known, but is not part of the general reader's background, and therefore needs to be explained. In the third section, we deploy the instruments of category theory to reach an understanding, on new grounds, of certain aspects of the Peircean continuum, the pragmaticist maxim, and the existential graphs. The leading ideas of category theory—synthetic constructions, genericity, freeness, systems of contrastation and gluing, and the interplays between global and local, universal and particular—enable us to highlight the conceptual richness and the contemporary relevance of the Peircean system. In the final section, we point out some of the open problems that arise from our category-theoretic reading of the system. In the spirit of Peirce's own work, we introduce a number of original diagrams to support the arguments advanced in this paper.

1. The Pragmaticist Maxim, Peirce's Continuum, The Existential Graphs

There are two opposed tendencies in the reception of Peirce's work: some, on the one hand, find in his writings a set of unsystematic reflections, diverse and imaginative as well as disordered; and some, on the other hand, find a complex architectonics, eminently unfinished, but with clear forces and supporting structures for its rational order. Without entering into this difficult debate, which will not concern us here, we will adopt at the outset the hypothesis that Peirce's work can be understood from an architectonic point of view, despite certain unavoidable lacunae in the system as a whole. Indeed, one of the goals of this paper is to provide new scaffoldings for the Peircean architectonic, with the aid of the instruments of category theory. We will see that this scaffolding is designed to be erected at some distance from the foundations (and consequently also far from the analytic trends of classical set theory) and that it throws into relief certain bundles of structures in the building. This allows us to call it a "castle in the air" (Murphey 1961:407), without thereby casting aspersions on the solidity and rigor that are required for its erection. Along the lines of Murphey's metaphor, the Peircean system deserves to be understood not as a construction with deep vertical foundations, like the early 20th-century Eiffel Tower, but rather as a typical early 21st-century structure, full of transverse horizontal bundles like Toyo Ito's Sendai Mediatheque, a "castle in the air" translucent and without foundations.

The **pragmaticist maxim**—as Peirce came to call it, to distinguish it from other interpretations (behaviorist, utilitarian and psychologistic)—was reformulated many times in the intellectual development of our author. The most famous statement is that of 1878, but those from 1903 and 1905 are more precise:

> Consider what effects which might conceivably have practical bearings we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object. (Peirce 1878:C5.402)

> Pragmatism is the principle that every theoretical judgement expressible in a sentence in the indicative mood is a confused form of thought whose only meaning, if it has any, lies in its tendency to enforce a corresponding practical maxim expressible as a conditional sentence having its apodosis in the imperative mood. (Peirce 1903m:C5.18)

> The entire intellectual purport of any symbol consists in the total of all general modes of rational conduct which, conditionally upon all the possible different circumstances, would ensue upon the acceptance of the symbol. (Peirce 1905e:C5.438)

The 1905 statement stresses that the knowledge of symbols is obtained by following certain "general modes" across a spectrum of "possible different circumstances." This modalization of the maxim (remarked in the odd repetition of "conceivability" in the 1878 statement) introduces into the Peircean system the problems of links between the *possible* contexts of interpretation that we can have for a given symbol. In turn, in the 1903 statement we see, on the one hand, that the practical maxim should be expressible as a conditional whose *necessary* consequent must be contrasted adequately, and, on the other hand, that any indicative theoretical judgment, within the *actual*, only can be specified by a series of diverse practices associated with the judgment.

Broadening these precepts to the general context of semiotics, we find that in order to know a given symbol (the context of the *actual*) we must run through the multiple contexts of interpretation that can interpret the sign (the context of the *possible*), and within each context, we must study the practical (imperative) consequents associated with each of those interpretations (the context of the *necessary*). In this process the *relations* between the possible contexts (situated in a *global* space) and the relations between the fragments of necessary contrastation (placed in a *local* space) take on a fundamental relevance; this underscores the conceptual importance of the logic of relations, which was systematized by Peirce himself. Thus the pragmaticist maxim shows that knowledge, seen as a logico-semiotic process, is preeminently contextual (as opposed to absolute), relational (as opposed to substantial), modal (as opposed to determinate), and synthetic (as opposed to analytic).

The maxim filters the world through three complex webs that enable us to differentiate the one in the many, and, inversely, to integrate the many in the one: the *modal* web already mentioned, a *representational* web and a *relational* web. Certainly, in addition to opening themselves to the world of the possible, the signs of the world must above all be capable of representation within the languages (linguistic or diagrammatic) that are used by communities of interpreters. The problems of representation (fidelity, distance, reflexivity, partiality, etc.) are therefore intimately linked with the *differentiation of the one in the many*: the reading of a single fact, or of a single concept, which is dispersed through multiple languages, through multiple "general modes" of grasping the information, and through multiple rules of organization, and of stratification, of the information.

However, one of the virtues of Peircean pragmatism and, in particular, of the fully modalized pragmaticist maxim, consists in making possible it to *reintegrate anew the many in the one*, thanks to the third—relational—web. Indeed, after decomposing a sign into subfragments within the several possible contexts of interpretation, the correlations between the fragments give rise to new forms of knowledge, which were hidden in the first perception of the sign. The pragmatic dimension stresses the *connection* of some possible correlations, discovering analogies and transferences between structural strata that were not discovered until the process of differentiation had been performed. Thus, although the maxim detects the fundamental importance of local interpretations, it also encourages the reconstruction of the global approaches by way of adequate *gluing* of the local. We will see in the third section how the tools of category theory give a great technical precision to these vague and general initial ideas. The pragmaticist maxim will accordingly be seen as a kind of abstract *differential and integral calculus*, which can be applied to the general theory of representations, i.e., to logic and semiotics as understood, in a more generic way, by Peirce.

In Figure 1 we present a diagrammatic schematization of the pragmaticist maxim, in which we condense synthetically the discussion up this point. This diagram will prove indispensable in the third section of this paper, in enabling us to understand in a natural way the structure of the maxim from the perspective afforded by category theory. Reading from left to right, the diagram shows an actual sign, which is represented in many ways (i.e., subdetermined) in possible contexts of interpretation, and whose necessary actions-reactions in every context give rise to partial understandings of the sign. The first process of differentiation is expressed by the terms 'pragmatic differentials' and 'modulations'; the latter term reminds us of the way in which the same motif can be extensively changed throughout the development of a musical composition. On the other hand, the process of proper reintegration of Peircean pragmatics is expressed by the terms 'pragmatic integral', 'correlations, gluings, transfers', which remind us of the desire to reunify what is fragmented, and whose precise technical underpinnings will be exhibited in the third section of this paper.

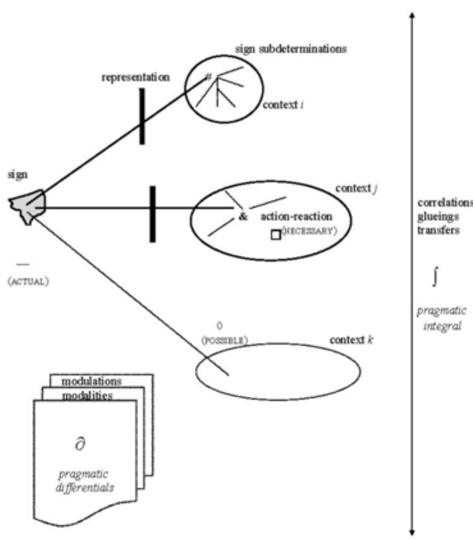

Figure 1. The (fully modalized) pragmaticism maxim

Underlying the good use of the pragmaticist maxim, applicable in theory to the broadest range of problems of knowledge, is a *hypothesis of continuity* between the world of phenomena and the spectrum of representations of those phenomena. That means that the relational links between the signs, and, in particular, the semiotic cascades between the Peircean interpreters, can be placed in a non-artificial generic ground. Peirce's *synechism* thus postulates a real operation of the continuum in nature and allows us to trust in a certain continuity that helps to bring together, in a natural way, phenomenology and logic (this is obvious in the Peircean classifications of sciences (Kent 1987), a matter we cannot go into here). On the other hand, from a merely intuitive point of view, the spectrum of modalities that emerges in the pragmaticist maxim immediately involves the postulation of a generic and abstract continuum that makes it possible to link the different modal gradations and correlations (a general intuition that Peirce will try to reproduce concretely with his "tinctures" in the existential graphs). A full modal and relational understanding of the pragmaticist maxim thus brings us to the Peircean continuum.

For a broad vision of the **Peircean continuum** we recommend Jérôme Havenel's doctoral thesis (Havenel 2006), the best available treatment of the subject, or our own monograph (Zalamea 2001). We notice here some of the features of the Peircean continuum that we will be able to discuss more fruitfully, in the third section of this paper, in category-theoretic terms. The Peircean continuum is an absolutely general *concept* which, in principle, should not be completely objectivized just in a single context of formalization. We have to do here with a really generic concept, intrinsically underlying every other general concept: "every general concept is, in reference to its individuals, strictly a continuum" (Peirce 1908d:C4.642). The Peircean continuum, as an unrealizedly free concept in the context of the wholly general and the possible, cannot therefore be fully delimited by any given collection: "no collection of individuals could ever be adequate to the extension of a concept in general" (Peirce 1905g:C5.526). Breaking free of the contexts of determination of the continuum—its partial extensions—and insisting on the intensionality of the continuum as a general concept—and not an object—Peirce obtains from the outset one of the most incisive features of his vision of the continuum. He obtains, in fact, an extremely important asymmetrization of Frege's principle of abstraction, since intension and extension, in certain cases like that of the continuum, are not necessarily logically equivalent.

In parallel with the recovery of the primacy of the concepts over the objects, Peirce insists that the continuum be understood synthetically, as a general whole that can*not* be reconstructed analytically as an internal sum of points:

> Across a line a collection of blades may come down simultaneously, and so long as the collection of blades is not so great that they merge into one another, owing to their supermultitude, they will cut the line up into as great a collection of pieces each of which will be a line,—just as completely a line as was the whole. This I say is the intuitional idea of a line with which the synthetic geometer really works,—his virtual hypothesis, whether he recognizes it or not; and I appeal to the scholars of this institution where geometry flourishes as all the world knows, to cast aside all analytical theories about lines, and looking at the matter from a synthetical point of view to make the mental experiment and say whether it is not true that the line refuses to be cut up into points by any discrete multitude of knives, however great. (Peirce 1897(?)-a)

As we will soon see, this synthetic reading of the continuum would be fully recovered by category theory in the closing decades of 20th century. It suffices here, for the moment, to say that the Peircean continuum, as a synthetic concept (as opposed to a Cantorian analytic object) incorporates a greater richness (indeterminate, general, vague) than the Cantorian object of the real numbers does, since what is conceptual includes a wider plurality than what is objectual, just as what is synthetic involves a wider distributive universality than what is analytic.

For our purposes, the most outstanding features of the Peircean continuum consist in three crucial *global* characteristics (genericity, reflexivity, modality), three subdeterminations of these characteristics (supermultitudinousness, inextensibility, plasticity) and four *local* methodologies (generic relationality, vagueness logic, neighborhood logic, *possibilia* surgery) that interweave, in local contexts, the global directions that articulate the general concept. Even before giving a more careful explanation of these characteristics, we can observe that the concept of continuity itself is better understood by way of the diagrammatic schematization of the pragmaticist maxim that we have proposed. Continuity is a *general protean* concept (MacLane 1992:120) which—like Proteus, the mythical sea-god who assumes many different forms—can be modeled in many different mathematical contexts. In Figure 2 the concept of continuity appears on the left, and its diverse counterparts on the right (axiomatic and formal counterparts, in set theory and in category theory; and conceptual alternatives in the Peircean framework). From a pragmaticist point of view, continuity would consist then in the *integral* of all its partial realizations, and,

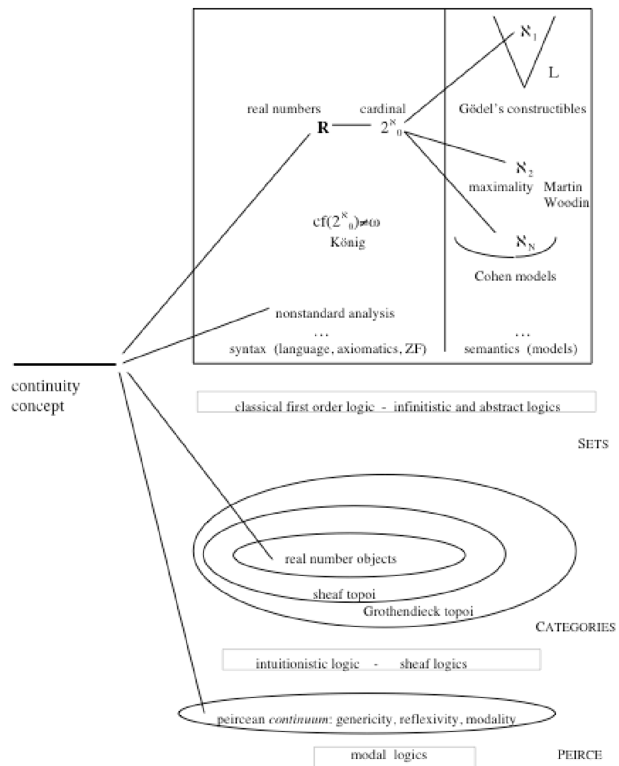

Figure 2. Continuity viewed through the pragmaticist maxim

in any case, it would not be possible to reduce it to the usual analytic Cantorian object within ZF (the context of interpretation in the upper right hand corner).

Perhaps the most outstanding feature of the Peircean continuum is its *general* character, with all the connotations and implications that the term includes. However, if we want to adapt ourselves a bit more to the language of contemporary mathematics, we should also use the term *genericity* as an equivalent substitute of *generality.* The general in Peirce involves many different nuances; all of them joined by the sign of the *free*—that which is far from particularizing, determining, existential, actual ties. The general is what lives in an extensive range of possibilities which are neither actualized nor determinate, and it is in contrast with the particular mode of the existent. Generality, as a law or regularity beyond the merely individual, as a fundamental element of reality beyond the merely named, falls into the Peircean category of thirdness and thus is naturally linked to the continuum. The Peircean continuum is thus a "general," which can contain everything potential—exceeding that which is determinate—and wherein certain modes of connection between the parts and the whole (the local and the global) are homogenized or regularized—exceeding and fusing the individual distinctions. The generic character of the Peircean continuum (its thirdness) is strictly related to the exceeding of the determinate and the actual (secondness), an exceeding in which the possible's elements of indeterminateness and chance (firstness)—those elements that free the existent of its particular features, and make generality accessible—are fundamental.

According to Peirce, the logic of relatives is the *natural filter* that permits to free and lay bare the active-reactive, and to melt it into in a broader general continuity, since it makes it possible to see the individual as a "degenerate" form of the relational, and the given as a degenerate form of the possible:

> Continuity is simply what generality becomes in the logic of relatives. (Peirce 1905h:C5.436)

> True continuity is perfect generality elevated to the mode of conception of the Logic of Relations. (Peirce 1905g:C5.528)

> Continuity is shown by the logic of relations to be nothing but a higher type of that which we know as generality. It is relational generality. (Peirce 1898c)

Peirce's motto—continuity = genericity via logic of relatives—is one of his most amazing intuitions. At first glance it appears to be obscure and cryptic but, as we will show in the third section of this paper, it is a brilliant abduction underlying the introduction of topological methods in logic and the demonstration (in the last decade of the 20th century) that many of the fundamental theorems of the logic of relatives are nothing but adequate theorems of continuity in the uniform topological space of elementary first-order classes. We believe that this Peircean abduction—perhaps one of the deepest expressions of Peirce's great logical refinement—is based on two previous and crucial logical experiments: on the one hand, the construction of his existential graphs (starting in 1896), in which the rules of logic are back-and-forth processes on the continuum of assertion (*discrete back-and-forth* for the calculus of propositions and *continuous back-and-forth* for the calculus of relatives); and on the other hand, the invention of his *infinitesimal relatives* (1870), with which Peirce discovers very interesting structural similarities between formal processes of differentiation (within the usual mathematical continuum) and operational processes of relativization (within a logical continuum that is much more general).

An immediate consequence of the genericity of the continuum is that the continuum must be supermultitudinous, i.e., that its size should be absolutely generic and should not be limited by any actually determinate size. The supermultitudinous character of the Peircean continuum shows that the Cantorian real line is no more than "the first embryon of continuity"— "an incipient cohesiveness, a germinal of continuity" (Peirce 1897(?)-a:N3.88). In fact, from the very beginning of their investigations, the pathways of Cantor and Peirce are opposite to one another; while Cantor and many of his successors in the 20th century try systematically to *delimit* the continuum, Peirce tries to *unlimit* it—to bring it nearer to a supermultitudinous continuum, not limited in size, truly generic in the transfinite, never totally actualizable.

Another consequence of this Peircean approach to the continuum—genuinely alternative to the Cantorian—is that the continuum cannot be constructed as a sum of existent beings (points) since, on the one hand, such a sum could be limited (and thus not be supermultitudinous) and, on the other hand, it could be made actual (and thus not inhabit the generic realm of pure possibilities). This alternative character of the Peircean continuum is also obtained by virtue of the *reflexivity* of the continuum: "a continuum is defined as something any part of which however small itself has parts of the same kind" (Peirce 1873a:W3.103). We will call the foregoing property of the continuum "reflexivity," since in a full continuum satisfying this reflection principle, the whole is *reflected in any* of its parts. Reflexivity implies that the continuum can*not* be composed of points, since the points—possessing no parts other than themselves—cannot possess parts that are similar to the whole. Thus reflexivity distinguishes the Peircean continuum from the Cantorian one, since the latter is composed of points and is not reflexive. In the Peircean continuum points as actualities disappear (they remain as *possibilities*) and these are replaced—in what is second, actual, active-reactive—by *neighborhoods* in which the continuum *flows.* We will see in the third section how this fluidity of the continuum is essential from a categorical point of view.

Here the property in virtue of which the continuum cannot be composed of points will be called *inextensibility.* As we have just pointed out, the reflexivity of the continuum implies its inextensibility (the Peircean continuum is reflexive and inextensible), or in other words, extensibility implies irreflexivity (the Cantorian continuum is extensible and irreflexive). The inextensibility of the Peircean continuum is closely related to another brilliant intuition of Peirce's, according to which no given class of numbers can completely codify the continuum. This limitation is a *natural* one showing that, in order to obtain a more precise understanding of the continuum, we must complement the program of the *classical arithmetization* of the real line with an alternative Peircean program: the construction of a *modal geometrization* of the continuum that, as we will see, the theory of categories has set up independently of Peirce.

The crucial *modalization* of Peirce's general thought begins at the time of his late readings of the Greek masters, around the middle of the decade 1880-1890 (Fisch 1986a:232). The Aristotelian influence—following his grasp of a broad range of possibilities applied to everything that is real—is soon perceived in the Peircean approach to the continuum, when he begins to present the continuum systematically as a complex modal *logos*:

> A continuum is a collection of so vast a multitude that in the whole universe of possibility there is not room for them to retain their distinct identities; but they become welded into one another. Thus the continuum is all that is possible, in whatever dimension it be continuous. (Peirce 1898e:R.160)

> You have then so crowded the field of possibility that the units of that aggregate lose their individual identity. It ceases to be a collection because it is now a continuum. . . . A truly continuous line is a line upon which there is room for any multitude of points whatsoever. Then the multitude or what corresponds to multitude of possible points,—exceeds all multitude. These points are pure possibilities. There is no such gath. On a continuous line there are not really any points at all. (Peirce 1903h:N3.388)

The great richness of real and general possibilities far exceeds the context of what exists, and constitutes a true continuum. The recursive Peircean contraposition between secondness and thirdness—a dialectic which grows and develops its potentiality in a permanent back-and-forth of reflections and iterations—is the contraposition between existence and being, between discontinuous mark and continuous flux, between point and neighborhood. In Peirce's view, while points exist as discontinuous marks which are *defined* with reference to the action-reaction of the scales of numbers over the continuum, the true and most permanent components of the continuum are *indefinite* or generic surroundings which are linked in the context of the possible without having actually to mark its borders. The metaphysical process that requires a general being in order to achieve the emergence of existence seems to be a process very similar to the genesis of the continuum: just like Brouwer, Peirce postulates the possibility of *previously* conceiving a global continuum ("perfect generality") on the basis of which, later, marks and systems of numbers are introduced, which locally imitate the general continuum.

The Peircean continuum, as a synthetic environment in which everything that is possible is glued, has to be a general place (*topos*), extremely flexible and plastic, homogeneous and without irregularities:

> The perfect third is plastic, relative and continuous. Every process, and whatever is continuous, involves thirdness. (Peirce 1886:W6.301)

The Peircean continuum is general, plastic, homogeneous, and regular, naturally permitting the *transit* of modalities, the fusion of the individualities, and the overlapping of the neighborhoods that shape it. The generic idea of *continuum flux* is behind those transits, fusions, and overlappings: ubiquitous osmotic processes which Peirce detects in the plasticity of protoplasm and of the human mind, and which—in a risky but deep abduction—give rise to cosmological universality (Peirce 1992b). The Peircean continuum—*generic and supermultitudinous, reflexive and inextensible, modal and plastic*—is thus the global conceptual medium within which explicative hierarchies can be naturally proposed, in order to limit those possible evolutions and local concretions embodying arbitrary notions of flux. We diagram these characteristics of the continuum in the following *double-sigma*; we lack space for the explanation of what we call "local methods," but we will discuss some of them in the third section, in order to connect this with the tools of category theory.

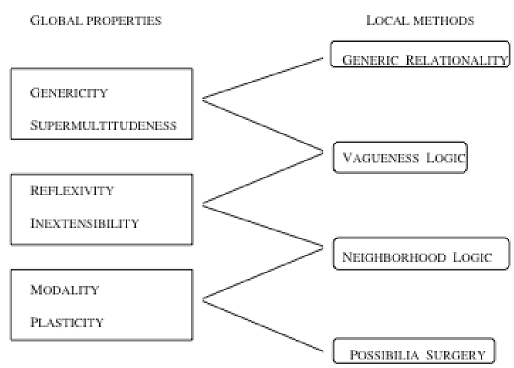

Figure 3. Main properties of Peirce's continuum

The idea of constructing reflections of the global in the local is an immediate consequence of the permanent transverse link between the structural arcs of the Peircean system—the pragmaticist maxim, the three categories, universal semiotics, the adjunction of the determinate and the indeterminate, the triadic classification of the sciences—a link that creates a natural hierarchy, in Peirce's great edifice, among the different floors and levels that communicate ceaselessly one with the other. The system of the **existential graphs** that Peirce considered to be his *chef d'oeuvre* (Peirce 1908b:N3.885) *iconically* reflects some of the most amazing and fruitful transverse crossings of his philosophical system. In fact, the ALPHA sheet of assertion, the continuous sheet on which the existential graphs are marked, stands as an icon reflecting the continuity of the real (thirdness), while the line of BETA identity, a continuous line that opens up the possibility of quantifying portions of reality, stands as an icon reflecting the continuity of existence (secondness). In this way, for instance, a real continuum, a third, can be thought, postulated and known, before we start imagining certain marks of existence (secondness). On the other hand, the fundamental rules underlying the radical novelty of the graphs—the rules of iteration/deiteration—are technical concretions of the large machinery of transverse osmosis proper to Peircean thought, incessantly interdisciplinary and often brilliantly original, thanks to the translations of concepts between diverse disciplines. Finally, the axioms for the graphs show that existence (the line of identity) is, simultaneously, a break of continuity in what is real and general (the blank sheet of assertion), as well as a continuous link in the particular (the extremes of the lines of identity). In this way, the lines of identity, continuous sub-reflections of the sheets of assertion, allow us to construct the passage from essence to existence when they are marked self-reflexively in the general continuum. The elementary axioms of the basic system of existential graphs thus support the idea—central in philosophy (Presocratics, Peirce, Heidegger)—that a first *self-reflection* of nothingness on itself is the spark that ignites the evolution of knowledge.

Peirce rightly pointed out that the existential graphs provided a full apology for pragmaticism. In fact, the existential graphs cover, in pragmatic fashion, the classical propositional calculus (existential graphs ALPHA) and classical first-order logic in a purely relational language (existential graphs BETA), as well as intermediate modal calculi, classical second order logic and the use of metalanguages (existential graphs GAMMA). Knowledge is constructed on the Peircean continuum (the general space of pure possibilities) by means of dual processes of action/reaction: insertion/extraction, iteration/deiteration, yes/no dialectics. The place of the Peircean continuum is represented by a blank sheet of assertion, in which some possible cuts are marked by means of precise rules of control, and information is introduced, eliminated and conveyed through them. The various marks that are inscribed in the sheet of assertion give rise to the evolution of logical information from the indeterminate to the determinate, due to the technical incorporation of a *formal graphical language, rules and axioms* (the doctoral dissertations (Roberts 1963) and (Zeman 1963) are still the best introductions to this topic).

A full apology for pragmaticism is obtained when we observe that the axiomatization of classical propositional calculus and of purely relational first-order classical logic—with the same rules, by means of the systems ALPHA and BETA—make explicit *technical* rules that are common and unnoticed in the current introductions to classical logic. In fact, the *same* rules detect, in the context of the ALPHA language, a propositional use, and in the extended context of the BETA language, a quantificational use: something that is incomprehensible and unimaginable for any student of logic educated in systems like Hilbert's. Thus—in accord with the pragmaticist maxim and with Peircean realism—the ALPHA and BETA calculi show that there is a nucleus, a *real general* underlying the logical transmission of information, a nucleus that, in certain contexts of symbolization, gives rise to the classical modes of connection, and that, in other contexts, gives rise to the classical modes of quantification. The rules of iteration/deiteration codify, in particular, the naturalness of the traditional logical operators; as we will see in the third section, this is not just a philosophical naturalness, but the technically well-defined naturalness of the fundamental information transmitters of category theory. The common roots of the connectives and the classical quantifiers are revealed in the same *programmatic, global, and general* action/reaction that in *diverse* contexts of symbolization gives rise to derivative rules, local and particular, proper to the context. This situation is a real revelation in the history of logic, which has not yet been appreciated; in any case, it constitutes, in a precise manner, the *only* known presentation of the classical calculi that makes global use of the same axiomatic rules to control the local grasp of connectives and quantifiers. The following diagram synthesizes the discussion:

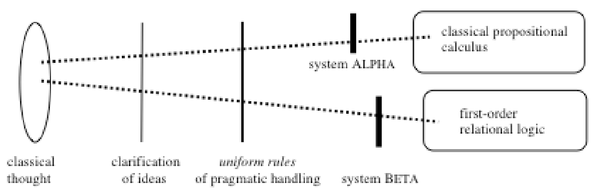

Figure 4. Existential graphs as an apology for pragmaticism

In turn, the apology for pragmaticism that is achieved with the existential graphs shows the coherence of synechism. Certainly, the rules, apparently discrete, of connectives and classical quantifiers correspond to one another *continuously* on a common generic background; their apparent differences are just contextual and can be seen as breaks in the underlying logical continuity. But even beyond the classical context, as we will show in the third section, we also have mathematical grounds for holding that synechism has a wider range of validity, comprising alternative forms—intuitionistic, category-theoretic, Peircean—for the logical continuum.

2. Some Fundamental Concepts of the Mathematical Theory of Categories

In this second section we present a very rapid introduction to some basic concepts of the mathematical theory of categories, which will be indispensable in the third section of this paper, and which do not usually form part of the background of the general reader. Our discussion is purely conceptual, leaving aside many technicalities that we cannot go into here; nonetheless, for a full comprehension of the bottom of the ideas which we will expound here, the reader will find it useful to consult more technical (but still generalist) works like (MacLane 1986) or (Lawvere and Schanuel 1997).

The mathematical theory of categories axiomatizes *areas* of mathematical practice, in accordance with the *structural* similarities of the objects in question and with the modes of *transmission* of information between these objects (it is here that the mathematical theory of categories is close to methodological and philosophical approximations sensitive to problems of *transference*, as in the case of Peircean pragmaticism, as we presented it in the previous section). As opposed to set theory, where objects are analyzed internally as aggregates of elements, the mathematical theory of categories (which from now on we will call "category theory") studies objects by way of their *external synthetic* behavior, due to the *relations* of the object with its context. The objects are like *black boxes*, which cannot be analyzed or broken into smaller interior sub-boxes, and which can be understood only by way of their *actions and reactions* with the surrounding medium. The modes of knowledge are then essentially relational: the ways in which the information transmitters behave in the context constitute the mathematical weaving in which knowledge advances.

A category is given as a class of objects (usually of the same structural type: combinatorial, ordered, algebraic, topological, differential, etc.) and a class of information transmitters ("morphisms") between the objects. Indeed, it is the morphisms, rather than the objects, whose nice properties constitute the true mathematical interest of the category.

A morphism is *universal* with respect to a given property if its behavior with respect to similar morphisms in the category possesses certain uniquely identifying characteristics which distinguish it within the categorical framework (for technical details see, for example (Lawvere and Schanuel 1997)). The basic notions of category-theory related to universality—those of *free object* and *adjointness*—respond to deep problems related to the search for relative archetypes and relative dialectics. In fact, after Gödel, the turn in mathematics toward problems of *relative consistency* (thus overcoming chimerical longings for absolute foundations) resulted in an explosion of diversity and differentiation in axiomatic mathematical theories, beyond a certain threshold of complexity. Within the resulting multiplicity in the broad, variable spectrum of the areas of mathematics, category theory managed to find some patterns of universality which facilitated processes of local unfolding and also the transcendence of concrete particulars. For instance, in a category, a free object is able to project itself into any object whatsoever taken from a sufficiently wide subclass of the category: it is thus a sort of primordial sign, embodied in all related contexts of interpretation. Hence, a sort of *relative universals* arose beyond relative localizations; these have given a new technical impetus to the classical notions of universality. Although it is no longer possible to presume that we are in a supposed absolute, nor to believe in uniform, stable concepts regarding space and time, category-theory has reshaped the notion of universality, making it suitable for a series of relative transferences of the universal/free/generic, in which *transition* is allowed, and in which at the same time it is possible to find remarkable *invariants* beyond it.

Thus, category-theory explores the structure of certain *generals* in a way similar to that of Peirce's late scholastic realism. Indeed, categorical thinking contemplates a dialectics between universal definitions in abstract categories (generic morphisms) and realizations of those universal definitions in concrete categories (structured set classes); moreover, within abstract categories, there may perfectly well be morphisms that are *real universals*, while at the same time not being existent (that is to say, they are not embodied in concrete categories: think, for example, about an initial object, readily definable in abstract categories, but which is not realized in the category of infinite sets, in which initial objects do not exist). In the range of pure possibilities, the pragmaticist maxim has to deal with the idea of universal concepts, logically correct, but which could possibly not turn out to be embodied in bounded contexts of existence (as, for example, in the case of the three Peircean categories: real universals that may not always adequately be realized in concrete existents within the bounded contexts). The mathematical theory of categories illuminates this kind of situation with a high degree of precision. The theory of categories has actually managed to effect the technical construction of a variety of entities, seemingly as elusive as those real universals with no existence, thanks to a very interesting dialectical process between the domains of actual mathematical practice (computational, algebraic, or differential structures, for example) and the possibility of abstract, universal definitions, still not realized in that practice. For instance, following current tendencies in universal algebra and abstract model theory, category-theory has been able to define really general notions of logics and of *relative truth universals*, as suitable invariants of given classes of logics. There accordingly remain a number of universal patterns beyond the multiplication of logical systems and varieties of truth. This is an example of the ways in which a certain welcome relativization need not imply a naïve, arbitrary dispersion of knowledge, something that the pragmaticist maxim defends as well.

Above and beyond the synthetic-relational merging of diverse areas of mathematics, category-theory is primarily interested in how these diverse areas can be compared with one another, and how the relevant mathematical information from one area can help to reconceptualize fragments from another, apparently remote, area. The *back* and *forth* of mathematical information, and the *control of those transfers of information* constitute one of the crucial rationales for categorical thinking. The mathematical entities which allow comparisons between two categories are called *functors*; and, to a great extent, a careful hierarchical study of a whole variety of functors is one of the central objectives of the theory. *Natural transformations* permit comparisons between functors, and a good global understanding of natural transformations brings out a great deal of local mathematical structure connected to the functors in question. Given a functor *F* between two categories, *C* and *D*, the existence of an *adjoint* functor *G* (optimal with respect to the construction of free objects throughout the natural structural inversion $Mor_D(X,FY)\sim Mor_C(GX,Y))$ turns out to be a completely ubiquitous situation in the mathematical world. Indeed, from the generic universal, *adjunctions* are exemplified by such concrete and apparently diverse constructions as free groups, polynomial rings, the bases of a vectorial space, discrete topologies, minima in an ordered set, implication, quantification and so on. A fine technical calculus of adjunctions brings about different *complex gluing systems* among mathematical objects, and allows a better understanding of "Mathematics' fundamental aporia" (Thom 1982:1133): the continuous/discrete opposition, also inescapable within Peirce's own system.

One of the basic initial results in category-theory is what is known as *Yoneda's lemma*, according to which every small category can be subsumed under an adequate category of functors (called *presheafs*, i.e., functors from the given category to the category of sets). This immersion of the initial category under a presheaf category brings about, in a *natural way* (in the beginning, a philosophically vague term that takes on a more precise technical sense as a consequence of natural transformations), several ideal entities which complete the universe, over a continuous background that remains hidden (as we will see in Section 3, this outcome provides new

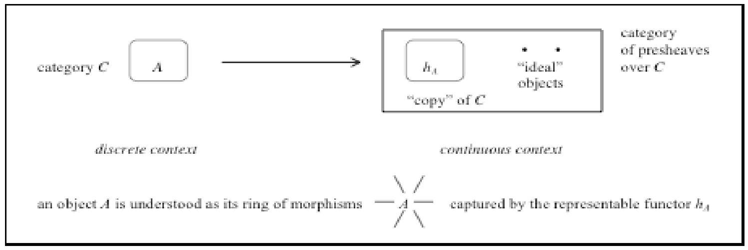

Figure 5. Yoneda's Lemma

support for Peirce's synechism). Figure 5 summarizes the situation we have been describing.

There are several forms of continuity behind Yoneda's lemma. *Representable* functors h_A symbolize all relations of the object *A* along with its context (category *C*), and capture through a single mathematical concept all of *A*'s relational information within the category. Representable functors preserve *all limits* (the notion of a diagram's limit is a categorical notion capable of being defined in a precise way) and, consequently, they are *continuous* (continuity is mathematically defined by means of an adequate limit preservation). The category of presheafs under which category *C* is subsumed is a *complete* category in the sense that it contains all limits, and it therefore behaves as a continuous natural environment. Furthermore, Yoneda's lemma is the basic tool for describing *classifier objects* in categories of presheafs and, hence, for comprehending the notions of internal logic in these categories. More generally, in the context of *topoi* (categories having good properties in the crossroads between set theory and algebraic geometry), Yoneda's lemma also allows the description of natural logical operators, which turn out to be intuitionistic in that context.

By applying these ideas, we will see in the next section how, from a pragmaticist point of view, we can amplify our conception of logic, opening ourselves to an array of *partial fluxes of truth* against a synechistic background, and we can explain, with a depth not suspected previously, the very *raison d'être* for the logical rules of iteration/deiteration in Peirce's existential graphs. The emergence of ideal objects (non-representable functors) as we attempt to capture a given reality (category *C*, or, equivalently, its representable functors), coincides with the peculiar mixture of realism and idealism characteristic of Peirce's philosophy. In fact, the linkage between the real and the ideal—inevitable and *demonstrably necessary* according to Yoneda's lemma— is a permanent, pervasive process in mathematical practice. Category-theory makes this linkage explicit, and allows interesting natural rapprochements with Peirce's system, as we will see in the following section.

3. Peirce's System from a Category-Theoretic Perspective

In this section the apparatus of the mathematical theory of categories is applied in order to re-examine and specify some aspects of the Peircean system as they were expounded in the first section. In particular, we will study five concrete respects in which the machinery of category-theory will help us to render more precise the concepts in question: (1) the problems of transference, linkage, and correlationality that are implicit in the pragmaticist maxim, and the way of understanding the maxim as a differential and integral abstract calculus; (2) categorical models in presheafs and sheafs, and the ways in which these models help us understand both problems of flux and fusion in connection with the Peircean continuum, as well as some links between logical systems and pragmatic truth rules associated with various forms of the continuum (classical, intuitionistic, category-theoretic); (3) bonds between continuity, genericity, and modality, and the ways of understanding them due to advances in topological model theory and intuitionistic logic against the general background provided by the categorical theory of topoi; (4) the specification of emergent problems regarding local methods associated with the Peircean continuum (generic relationality, linked to Freyd's allegories; neighborhood logic, linked to sheaf logic; and *possibilia* surgery linked to bi-modalities in algebras of topoi); (5) construction of new categorical models for alpha and beta existential graphs, and their extension to intuitionistic logic models, sheaf logic, and complex variables, which opens new and unsuspected doors for Peirce's graphs in mathematics.

3.1. The pragmaticist maxim as an abstract differential and integral calculus

Let us return to the pragmaticist maxim, entirely modalized, and its diagrammatic expression in Figure 1. From a categorical point of view, the diagram situates, above all, a sign within an abstract category to the left; and, to the right, the same sign partially embedded in diverse concrete categories. Diverse "modulations" and "pragmatic differentials" allow the delimitation of the singular, abstract, general sign, converting it into something multiple, concrete, and particular. This is something that, in category-theory, is achieved by means of the diverse functors at stake, which, depending on the axiomatic richness of each categorical environment on the right side, embody the general concepts in mathematical objects of more or less structural richness. This very first process of specialization towards the particular, of concretion of the general, of differentiation of the One, may therefore be understood as an abstract *differential calculus*, taken in the most natural sense possible: in order to study a sign, one must first introduce its *differential variations* in adequate contexts of interpretation. But, both from the point of view of the pragmaticist maxim, and the point of view of category-theory, this is but the very first step of a *pendular* dialectical process.

Indeed, once we know the variations of the sign/concept/object, the pragmaticist maxim urges us to *reintegrate* those diverse partial bodies of information into a whole that constitutes the knowledge of the sign itself. Category-theory also tends to show that, beyond concrete knowledge of certain mathematical objects, there are strong functorial correlations between them (in particular, adjunctions), which are what really inform us deeply regarding the concepts in question. In both approaches, we are urged to complete our ways of knowing, following the guidelines of an abstract *integral calculus*, the pendular counterpart of differentiation, which allows us to detect a certain closeness among several concrete particulars that seemed remote, but which respond to a natural closeness against a *prima facie* imperceptible structural background. Two paradigmatic examples illustrate this situation: *from the pragmaticist perspective*, rules of iteration/deiteration that permit us to bring together the discrete propositional (alpha) and the continuous quantificational (beta) against the structural background of existential graphs; *from a category-theoretic perspective*, the notion of a free object which allows us to bring together a ring of polynomials (a *sine qua non* object for the discrete control of algebraic structures) and an initial topology (a partial object in the continuous control of differentiable structures) against the structural background of adjunctions.

The integral/differential *back and forth*, found in the pragmaticist maxim as well as in category theory, may in fact be partially formalized in a second order Gamma version of the Existential Graphs, and thanks to this formalization, it can serve as the starting point of a *local proof for pragmaticism* in the language of those graphs. The formalization captures the essentials from Figure 1—diagrammatically, in Gamma language—in a second-order modal statement; this statement turns out to be demonstrable in part within an intermediary modal system, where the *possibility of the necessity of p* implies *p* (Nubiola and Zalamea 2007). It can be argued that a modal system including the just-stated law ("the possibly necessary is actualized") is a correct system for approximating Peircean scholastic realism, which provides us with a strong linkage between the pragmaticist maxim (Figure 1), the Existential Graphs (upon their underlying continuum), a partial proof for pragmaticism, and Peircean realism.

Vertical links on the right of Figure 1—denoting "correlations, gluings, transfers," and placed below the general sign of the "pragmatic integral" ($\int$)—codify some of the most original contributions both of the broad modal pragmaticism defended here, and of category theory. As we will further see in subsection 3.2, sheaf logic, which to a great extent underlies categorical thinking, enables us to define such notions as "gluing" and "transfer" with an appropriate degree of precision. Nevertheless, without going further into the technical details, one of the main problems of both Philosophy and Mathematics is how to partially glue certain classes of information-fragments into a coherent whole. Peirce's entire system, along with its architectonic transferences among the sciences—and in particular with its pragmaticist maxim—seems to be constructed in order to resolve, to a large extent, these kinds of problems. Indeed, from an analogical and a metaphorical point of view (indispensable in Peirce, as well as in category-theory, despite the fact that metaphoric concepts tend to be overlooked and disregarded by the most extreme trends in analytic philosophy), all of Peirce's system seems to be governed by a complex structural ordinance of the prefix *TRANS*. Everything in Peirce is *knowledge transit* and an attempt to understand relative dynamisms in knowledge, from experiments upon *relative differentials* in his early years, to *dynamical interpretants* in his later years. In an incessant delimitation and sub-definition of modes of transit, the transverse structures of the Peircean system (the three categories, the pragmaticist maxim, the classification of the sciences, semeiotic), as well as those precise logical tools that support them (relatives, the Peircean continuum, the Existential Graphs), provide many filters in order to control the movement and osmoses between concepts.

3.2. Intentionality, Sheafs, and Continuity

From the point of view of the axiomatic bases required to recognize an intensional, inextensible, and generic continuum as the Peircean continuum, Zermelo's local axiom of separation (the basis of Zermelo Fraenkel analytical set theory) appears to be an excessively demanding postulate. A stricter asymmetrization between intensionality and extensionality, even at a local level, could prove fruitful. The pre-eminence of intensionality would provide, in the first place, an important support to the inextensibility of the continuum. Actually, as a consequence of the asymmetrization of Zermelo's axiom of separation, only certain classes of formulae would determine classes, and the *a priori* "existence" of "points" could be eliminated: there would not always be singleton sets $\{x\}$ and only in certain specific, constructible cases could they be actualized. At the same time, permitting the manipulation of contradictory intensional domains (in the realm of potentiality) without dealing with the associated contradictory extensional classes (in the realm of actuality) that trivialize the system, would make it possible to have greater flexibility in a generic approach—free from actual hindrances—to the continuum. It is important to notice that (Bénabou 1992), (Nelson 1992) and (Thom 1992) also hold that the extension-intension symmetry, a credo of contemporary standard Mathematics, ought to be collapsed.

Both Peirce's version of the continuum and the general mathematical theory of categories are directed toward an intensional reading of the continuum. Indeed, category theory—constructed as a generic environment for a transverse study of information flows among different mathematical structures, a reticular environment weaved by a synthetic comparison of diverse universal properties—furnishes a paradigm in which the objects of study are defined intensionally, and in which clearly alternative knowledge methods are adopted: characterization of entities through morphisms, and *not* through elements; vision and creation through processes of synthesis, and not analysis; knowledge that is relational, contextual, external, and *not* combinatorial, isolated, internal.

Important examples of those intensional constructions are the categories of presheaves and sheaves that provide (among other things) partial models for the Peircean continuum. Categories of presheaves arise in a natural way with Yoneda's lemma, as we have seen in Section 2. On the other hand, a *sheaf* is a *presheaf* that allows the gluing, with generic elements, of diverse *compatible* collections of information codified in the presheaf. Certain categories of presheaves are useful for the partial actualization of some aspects of the genericity and inextensibility of the Peircean continuum. One of those environments, capable of creating a *synthetic geometry of the continuum*, is the category C of L^{op} functors in the category of sets, where L is the category of formal varieties C^{∞} (Moerdijk and Reyes 1991). A copy in C (via Yoneda) of the real Cantorian line, called the *smooth line*, is suitable for approximating the *fluxes* of the Peircean continuum: it is not Archimedean, it possesses infinitesimals, it is not determinable by points, it possesses a generic copy (non-standard) of the natural numbers. This shows that a copy (more precisely: an interpretant) of an incomplete concept, in a given context, may become naturally completed in another richer context, something perfectly in agreement with the pragmaticist maxim and Peirce's semeiotic (carefully categorized in (Marty 1990)).

On the other hand, different internal models in categories of *sheaves* allow for the separation (the making of a prescision—prescinding, as Peirce calls it) of certain properties *fused* in the real Cantorian line (R), showing, in a different way, that the latter contains too much superfluous structure and is not sufficiently generic. Indeed, in every sheaf category $Sh(O(T))$ over a topological space $(T,O(T))$, various copies of the real Cantorian line can be constructed (Troelstra and van Dalen 1988), and in the particular case of the category $Sh(O(R))$, the copies contructed by means of Dedekind cuts (R^d) and by means of Cauchy sequences (R^c) are distinct, with closure properties neatly detached from an intuitionistic point of view (R^c is real-closed, R^d is not). Even though intuitionistic models in sheaves do not seem to be more than "first embryons" of continuity, the *sheaf* logic underlying those models technically provides a finer handling of genericity and neighborhood logic, both of which are present in the Peircean continuum.

Sheaf logic, proposed in a highly flexible, fruitful form in (Caicedo 1995b), includes a wide range of intermediate logics between intuitionistic logic and classical logic. Given a topological space, Caicedo defines a natural local forcing on open sets, which renders precise—with all the accuracy of contemporary mathematical logic, and independently of Peirce—the fundamental Peircean idea that truth is generically *local* and not punctual: something is valid at a point if and only if it is valid in a neighborhood around the point. Sheaf logic confers a precise *coherence* on many of Peirce's ideas. Caicedo's results do a good job of handling some of the problematics surrounding genericity and neighborhood logic, and opens up fascinating new perspectives: the construction of a theory of *generic models* allows us to obtain—uniformly, as simple corollaries of the construction of generic structures in suitable sheaves—the fundamental theorems of classical model theory (completeness, compactness, omitted types, Los's theorem for ultraproducts, set-theoretic forcing), while at the same time the study of interconnections between the usual punctual (Tarskian) semantics and local sheaf semantics allows us to reconstruct classical truth, in the sheaf fibers, as a (natural, pragmatic) *limit* of intuitionistic truth, characteristic of its global sections. In his elaboration of sheaf logic—which he constructs in an intermediate layer between Kripke's models and Grothendieck's topoi, taking advantage of the many concrete examples of the former and the general abstract concepts of the latter—Caicedo works in the crossroads of algebraic, geometrical, topological, and logical techniques. The *back* and *forth* between the generic and the concrete, as well as the *transverse crossing* techniques, show that in his very method of research (over and above the similarities in objectives) Caicedo is very close to Peirce.

Caicedo's contributions show that—just as Newtonian mechanics can be seen as a limit in Einstein's relativity, and Euclidean space can be seen as a limit in Riemannian geometry—classical logic should be understood as a *limit in sheaf logic*. The awareness of this bordering situation gives rise to two innovative explanations of great depth: on the one hand, it explains the prominence classical logic has achieved in its historical development during the 20th century, since it turns out to be the kind of *natural* logic that best suits the "Cantorian program"—the construction of Mathematics as a punctual sum of ideal actualizations, in a static, Platonic context; on the other hand, it opens up vast perspectives for the *continuum* of intermediate logical gradations between intuitionistic and classical logic, and singles out sheaf logic as the *natural* logic that best suits what we will later call a sort of "Peircean program" for Mathematics: a re-construction of Mathematics as a relational differential of real possibilities, in an evolutionary, Aristotelian context.

3.3. Topological Logic, Topoi, Genericity, Continuity and Modalities

Other findings of Caicedo's—on global continuous operations that codify structural properties of extensions of first order classical logic (Caicedo 1995a)—yield an illuminating perspective on Peirce's fundamental weaving of continuity, and the logic of relations. Applying topological methods in model theory, Caicedo shows that general axioms in abstract logics coincide precisely with continuity requirements on certain algebraic operations between model spaces, and he establishes an extensive list of correspondences between topological and logical properties, many of them based on the systematic examination of the *uniform* continuity of natural operations between structures. Caicedo's theorems, according to which substantial portions of abstract model theory correspond to precise topological phenomena, may be interpreted in several different ways in order to elucidate the apparently cryptic Peircean motto: *continuity = genericity via the logic of relatives.* In fact, on the one hand, it may be observed that the *general* (axioms of abstract model theory), *filtered* through the web of the logic of relatives (classical first-order logic), yields a natural *continuum*—a uniform topological space by means of localized elementary equivalence (Caicedo 1995a:266)—and uniform continuity of operations in that web: projections, expansions, restrictions, products, quotients, exponents (Caicedo 1995a:273). On the other hand, the fact that closure under relativizations in an abstract logic is *equivalent* to the comparison of adequate uniform topologies in model spaces (Caicedo 1995a:276)—simultaneously demarcating and detaching the validity or invalidity of many logical transfers—shows that the relative and the contrasting of the continuous can coincide at the highest level of abstraction, *free* and *general*, in the broad sense that these terms acquire in the dyad of category theory and the Peircean system.

Various developments in pure mathematics—whether in the theory of categories, sheaf logic or topological logic—help to make some of Peirce's insights into the continuum much more precise, especially with regard to its global features of genericity and reflexivity (and, consequently, inextensibility). On the other hand, mathematics has not yet provided proper models for understanding the *modal* supermultitudinous environment of the Peircean continuum, where the *whole* universe of *possibilia* should be located. In this collection, Philip Ehrlich has proposed a fascinating and ingenious construction of the supermultitudinous, and of the *ordinal punctual density* of a continuum—which is very close to the Peircean one—within extensions of NBG set theory; but, even though it is the finest model provided so far in order to capture the Peircean supermultitudinous, it is a model that loses the fundamental *modal* intuition that motivated the late Peirce. From the point of view of the theory of categories, an alternative, suitable tool to approach modalities has been proposed in (Reyes and Zolfaghari 1996), thanks to certain natural algebraic structures that emerge in topoi. In fact, the subobject classifier in *any* topos is provided with a natural Heyting-algebraic structure, where, as a consequence of Yoneda's lemma, the internal logical operators of the topos can be described. Surprisingly, it seems that these operators obey intuitionistic laws, in which there is no law of excluded middle, and, where the double negation operator (¬¬) does not coincide with a simple "yes" in the actual, but with a sort of "dense-yes" in the future. Amplifying this situation, Reyes and Zolfaghari have formalized some of Lawvere's pioneering intuitions on *boundary* abstract operators, and they have shown that Heyting's algebras in topoi possess in fact a *dual* structure of bi-algebras, where plenty of modal operators appear to be *limits* (continua) of natural iterations of available differentiation and negation operators in bi-algebras. As a consequence, in particular, every presheaf topos possesses an infinite hierarchy of intermediate modalities. Thus, modalities appear in a much more ubiquitous form than possibly imagined at first glance, and they are structurally interrelated to forms of continuity in the topoi. Infinite hierarchies of modalities in presheaf topoi (ubiquitous in all regions of mathematics) may then help us to model the enormous Peircean universe of *possibilia*, and, furthermore, to clear the way for a technical understanding of the complex interrelations between continuity and modality.

3.4. Local methods related to Peirce's continuum

In the first section of this paper we have already pointed out that some of the local methods connected to the basic features of the Peircean continuum (genericity, reflexivity, modality) could be described in terms of the following techniques: generic relationality, neighborhood logic, *possibilia* surgery, vagueness logic. In what follows, the first three methods will be briefly introduced, and they will be discussed in terms of category-theory. In Figure 3, these properties of the Peircean continuum were codified in a *double-sigma* that metaphorically invited us to think in terms of Watson and Crick's double-helix, an intertwined double spiral staircase where genetic information is stored. Just as the double-helix codifies the essential part of the secrets of living beings, the double-sigma is intended to codify the essential part of the secrets of the continuum. In fact, in the fourth section we will see how this double-sigma brings about an ambitious program for re-examining the Peircean continuum, a re-examination that might be extended into other branches of philosophy and mathematics.

As for the genericity of the continuum, Peirce points out that the mode of connection among parts ought to be captured in all its generality, and only by means of a genuine—non-dyadic—triadic relation, thus clearing the way for the study of *generic triadic relations* that could be closely related to general modes of gluing and contiguity:

> No perfect continuum can be defined by a dyadic relation. But if we take instead a triadic relation, and say *A* is r to *B* for *C*, say, to fix our ideas, that proceeding from *A* in a particular way, say to the right, you reach *B* before *C*, it is quite evident that a continuum will result like a self-returning line with no discontinuity whatever . . . (Peirce 1898c:C6.188)
>
> The attraction of one particle for another acts through continuous Time and Space, both of which are of triadic constitution . . . . The dyadic action is not the whole action; and the whole action is, in a way, triadic. (Peirce 1908a:C6.330)

These statements show that Peirce tries to find accurate manifestations of the global in the local: the continuum—which, in its perfect generality is one of the most achieved global forms of Thirdness—must also embody a genuinely triadic mode of connection in the constitution of its local fragments. The type of generic triadic relationality Peirce was looking for might well be subsumed under Peter Freyd's *allegory theory*, which we now proceed to review.

Freyd's allegories constitute an axiomatic environment for the study of abstract categories of relations in which morphisms intend to capture, not functions between objects, but all kinds of arbitrary relations. Allegories allow us to construct the pathway from the structured to the *structure-free*, in the sense of the type of genericity that Peirce himself was looking for. Indeed, by means a procedure ubiquitous in categorical logic, (Freyd and Scedrov 1990) show that, starting from pure theories of types with certain structural properties (regularity, coherence, first-order, higher-order), *free* categories that reflect the structural properties given at the beginning (regular categories, pre-logos, logos and topoi) can be *uniformly* constructed —by means of a completely controlled architectonic hierarchy.

In obtaining free categories, we obtain the barest possible categories that can be reflected into any *other* category with similar properties: thus, Freyd manages to construct the *initial archetypes* of mathematical theorization. Within Freyd's allegorical environment, the generic relationality we were seeking earlier for the Peircean continuum would then correspond to the search for an archetypal triadic relation in a free category (called a *classifier topos*, if some further restrictions are added) associated with the continuum. This is actually a work in progress. Even though diverse *classifier* *topoi* for algebraic and geometrical sub-theories of the mathematical network have been obtained so far, *classifier topoi* for topological theories seem to surpass our current developments.

The Peircean continuum is constituted by real contexts and neighborhoods, modes of connection and fusion of *possibilia*, transactions and overlappings among each other. Upon this continuum only ideal points are marked—discontinuities and gaps along the actual—in order to build scales of comparison and to facilitate the calculus. The apparent strangeness of synthesising the real and the possible, the ideal and the actual, is just another one of the radical innovations brought about by Peirce's philosophy. In fact, the actual, the given, the punctual, the present, the instantaneous are no more than *ideal limits*: limits in contexts of possibility that really contain those marks of actuality, those intractable points, those passing presents, those impalpable instants. It is understandable, then, that Peirce insisted so much upon the fact that the continuum should be carefully studied along with its inextensibility through *neighborhood logic*, a variety of logic that allows us to study modes of connection between *real contexts*, a border logic of the intermediate, a logic that is to a great extent—as would later be discovered in the 20th century—different from a *punctual* logic:

> The point of time or space is nothing but the ideal limit towards which we approach indefinitely close without ever reaching it in dividing time or space. To assert that something is true of a point is only to say that it is true of times and spaces however small, or else that it is more and more nearly true the smaller the time or space, and as little as we please from being true of a sufficiently small interval. . . . And so nothing is true of a point which is not at least on the limit of what is true for spaces and times. (Peirce 1873a)

> A drop of ink has fallen upon the paper and I have walled it round. Now every point of the area within the walls is either black or white; and no point is both black and white. That is plain. The black is, however, all in one spot or blot; it is within bounds. There is a line of demarcation between the black and the white. Now I ask about the points of this line, are they black or white? Why one more than the other? Are they (A) both black and white or (B) neither black nor white? Why A more than B, or B more than A? It is certainly true, First, that every point of the area is either black or white, Second, that no point is both black and white, Third, that the points of the boundary are no more white than black, and no more black than white. The logical conclusion from these three propositions is that the points of the boundary do not exist. That is, they do not exist in such a sense as to have entirely determinate characters attributed to them for such reasons as have operated to produce the above premisses. This leaves us to reflect that it is only as they are connected together into a continuous surface that the points are colored; taken singly, they have no color, and are neither black nor white, none of them. Let us then try putting "neighboring part" for point. Every part of the surface is either black or white.

> No part is both black and white. The parts on the boundary are no more white than black, and no more black than white. The conclusion is that the parts near the boundary are half black and half white. This, however (owing to the curvature of the boundary), is not exactly true unless we mean the parts in the immediate neighborhood of the boundary. These are the parts we have described. They are the parts which must be considered if we attempt to state the properties at precise points of a surface, these points being considered, as they must be, in their connection of continuity. One begins to see that the phrase "immediate neighborhood," which at first blush strikes one as almost a contradiction in terms, is, after all, a very happy one. (Peirce 1893d:C4.127).

Peirce's arguments show that talking about "points" at the boundary of a drop of ink is just an ideal postulate; on the sheet there really exist only colored neighbourhoods of three specific types: black, white, or black *and* white environments. Boundary points are characterized as those ideal entities that can only be approximated by environments of the third type. Therefore, neighbourhood logic, or logic of continuous colors, immediately embeds elemental forms of triadicity and surmounts the law of the excluded middle. It is reasonable, then, that Peirce was the first modern logician to construct intermediate truth tables for connectives of a three valued logic.

In the Peircean continuum, neighbourhoods are environments of the possible, in which a supermultitude of potential points are accumulated. It is of the utmost importance, then, to construct a local surgery for the geometry of those environments of the possible, a local surgery that must incorporate similar techniques to Whitney's surgery in differential topology, with which germs of possibilities could be systematically glued and developed. This *possibilia surgery*—yet to be developed, but nonetheless implicit in Peirce's thought—should be naturally related to Thom's cobordism techniques (a *generic cobordism* might be part of the generic ground of Thirdness), with their recourse to a qualitatively homogeneous topos (Thom 1992), similar in many respects to the Peircean continuum, and with the natural calculus of bi-modalities emerging in the algebras of a topos.

For example, several mathematical techniques are already available in the context of an algebraic geometry-oriented surgery (Levine and Morel 2007). Levine and Morel describe their work as a transposition of Quillen's techniques for cobordism in differential varieties, now transposed into the context of Grothendiek's *Motives*. The process of mathematical thinking exhibited here might be considered as typically Peircean (and, of course, category-theoretic): ideas of surgery (cobordism) are first understood in differential contexts (Thom, Quillen), then they are divested of all excess structure (Grothendiek), and they are finally studied in the most generic context (motives). In fact, when motives are reached, it is possible to arrive at a very deep generality, since motives constitute a sort of initial archetype for cohomologies, in their turn understood as invariant webs of higher genericity (and, therefore, less complexity) in order to grasp a mathematical area with more particularized concrete structures (and, therefore, of higher complexity). For instance, that is precisely the case of differential and topological subcategories, which, having a strong mathematical structure, are to be grasped through proper algebraic cohomologies of lesser mathematical complexity. The back and forth between particulars and generals, along with controls within a complex hierarchy of transits and obstructions, is a typically Peircean procedure that can be observed here and that has been turned into a genuine catalyst for development by the mathematical theory of categories.

3.5. Category-theoretic models for existential graphs

As we have seen in the first section, existential graphs do provide a sort of *generic nucleus* underlying the rules of logical transference and giving rise, on the one hand, to classical propositional calculus (ALPHA system), and, on the other hand, to first order classical logic based upon a relational language (BETA system). It is not difficult to recognize, then, that this generic nucleus could actually be well-defined in terms of the mathematical theory of categories—a sufficiently well-stocked toolbox to deal with instances of universality and genericity within well-defined mathematical contexts.

Indeed, (Brady and Trimble 2000b) have proposed a categorization of system ALPHA within the context of *monoidal categories* (categories with a tensorial functor, in which it is possible to define an abstract notion of a monoid in a natural way: ubiquitous categories that appear through free word categories, endofunctors, modules, etc.), and have showed that (i) every ALPHA graph gives rise to an algebraic operation in a *Lawvere algebraic theory* (a particular case of monoidal category); and (ii) system ALPHA's deduction rules are factorized through *functorial strengths* (natural transformations introduced in order to resolve coherence problems in abstract categories, which have emerged then in such different contexts as Riemannian geometry, weak forces in subatomic physics, counting in Girard's linear logic, etc.).

On the other hand, (Brady and Trimble 2000a) have indicated how to represent BETA graphs by means of a relational categorical calculus associated to a first order categorical theory. This representation uses neither the general framework of Freyd's allegories nor Lawvere's hyperdoctrines, but an intermediate level of representation, with logical functors that create quantifiers and that verify Beck-Chevalley's condition (an abstract categorical mode of dealing with free variables in standard logic). The free category of relations is in correspondence with a (monoidal) category of string diagrams, in the style of (Joyal and Street 1991), and the desired representation within the system is provided by a proper quotient through a congruence that codifies the BETA system.

Brady and Trimble construct their representations in restricted cases of *-autonomous categories (generalized models of Boolean algebras). Nevertheless, functorial equations of commutation for crucial rules of *iteration* and *deiteration* in graphs are also *intuitionistically* valid. The elimination of double cuts, which is valid in classical logic, but not in intuitionistic logic (since double negation is *not equivalent* to affirmation, as we saw before), does not need to be used here. Therefore, it is possible to find a whole new panorama of *natural intuitionistic* comprehensions of existential graphs, within monoidal categories, in which double negation operators are *genuinely* alternative operators; that is to say, something distinguished from mere identity. Indeed, diverse systems of intuitionistic existential graphs, along with *new diagrammatic connectives* irreducible to classical implication and negation, are actually being constructed by (Oostra 2008), and the exploration of these systems from a category-theoretic point of view has just been started by (Zalamea 2008).

Through Yoneda's lemma (whose ubiquitous efficacy has been remarked several times in this paper), a *connective in a topos* can be defined as an adequate natural transformation in the light of Heyting's algebra of sub-objects of the topos. In this case, Peircean rules of *iteration* and *deiteration* appear to be *precisely* the technical conditions required for assuring naturalness in transformation; and, furthermore, they also constitute *precisely* the very same conditions for an arbitrary connective to be technically characterized as an *intuitionistic* connective (Caicedo and Cignoli 2001). These diverse links (particularly guided by forms of iteration/deiteration) among categories, intuitionism, and existential graphs—links which are located against the very same topological framework as Peirce's (Havenel 2006)—show the richness of logico-topological intuitions codified in the graphs. It is a much more thorough, complex panorama than possibly expected, with a very ample mathematical potential.

4. Some Open Trends

In this present section we are going to remark, very briefly, some open trends regarding the exploration of the Peircean system by means of the toolkit provided by the mathematical theory of categories.

4.1. Continuum Pragmae

To construct a categorical topics capable of studying in a systematic way the synthetic global correlations among knowledge places; and to construct a modal geometry capable of studying modes of local connection among those places and detecting their modal invariants (Figure 6).

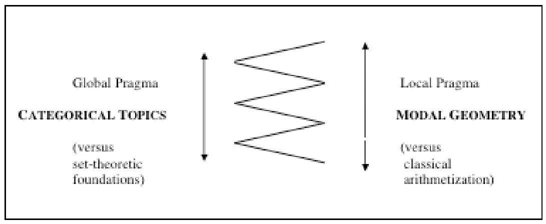

Figure 6. Pragmae related to Peirce's continuum

Coming from a vertical—pragmatic—reading of the double-sigma (Figure 3), these are two vague, undetermined programs (in the Peircean sense), but which nonetheless must provide *alternatives* for analytical philosophy, whose origins were drastically determined by foundations, sets, and classical logic, and whose case studies in the philosophy of mathematics are frequently reduced to elementary arithmetical and propositional considerations, without contemplating all the multivalent richness of contemporary mathematical thinking (category-theoretic, in particular). To make these alternatives more precise: the *continuum pragmae* propose a reconstruction of Mathematics as a *relational differential of real possibilities in an evolutionary, Aristotelian context*. In this last phrase every term stands in contrast to a corresponding *pendular* concept within the analytical account of classical set theory. In fact, from the point of view of the mathematical theory of categories, Mathematics turns out to be:

- *Evolutionary* (objects in a topoi are, basically, sets developed over time), a fact that stands in contrast to the classical belief that Mathematics is static and rigid through time;
- *Aristotelian* (categories propose a web of contrasts between naturalness and artificiality, always regarding a hierarchy of real obstructions in the mathematical world), something opposed to the Platonic view of Mathematics as placed in the ideal world;
- *Relational differential* (categories uncover a relational unity beyond difference, but they assume this differentiability as the very catalyst of mathematical thinking), something contrasted to classical reductionisms in set theory;

- *Modal* (categories configure a web of incessant translations, which may be understood as webs of representations and interpretations open to modulation/modalization), a fact that stands in contrast to the classical uniform reconstruction of Mathematics through the actual Cantorian infinite.

4.2. Allegorical Program for the Continuum

To construct models for Peirce's and Thom's archetypal continuum, from certain differentiable structures, and liberating them toward the real generic by means of Freyd's allegoric machinery.

It will be necessary to simplify here and to understand, in an abstract methodological context, the new ideas that are continually emerging in advanced mathematical techniques, such as algebraic cobordism and motivic cohomology (see Section 3.4). The whole mathematical tendency to look for initial archetypes (classifier topoi, free allegories, motives) must be able to help with the Peircean pursuit of an initial generic continuum, which may be *projected* into different partial contexts of continuity (such as, for example, the sheet of assertion in existential graphs).

4.3. "Mainstream" Mathematical Models for Existential Graphs

To construct relevant mathematical models for existential graphs within three "mainstream" lines of mathematical research, in order to make existential graphs something closer to important questions in Mathematics: complex variables (Zalamea 2003), sheaf logic (Caicedo 1995b), or monoidal general categories (Brady and Trimble 2000b; Brady and Trimble 2000a).

It is no longer possible to continue thinking about graphs as a mere *language*, or as a peculiar diagrammatic syntax; and it is simply essential to incorporate its semantic and pragmatic complexity within the main boundaries of Mathematics. The *natural* transit between intuitionistic logic, its topological models, and Peirce's logico-topological thought ought to be made explicit and developed completely. Connections among intuitionistic existential graphs, sheaf logic, complex fibres, and Riemann's surfaces (which may be grasped by GAMMA books of sheets of assertion), might be able to radically transform, not only our eccentric, isolated perception of graphs, but also our whole comprehension of nuclear—and yet unexplored—issues in Mathematics, such as the intrinsic logic of the complex variable.

Notes

1. David Villena and Ignacio Redondo have done a beautiful job of translating a not always easy paper. Daniel Campos and Matthew Moore have smoothed out both the English and also some conceptual obstructions. Moore advised me on an overall rewriting, which has helped to simplify the presentation greatly. To all, my warm thanks. Of course, I am to be held responsible for all remaining obscurities.